that the solution studied here is the major term of the "inner expansion" of the exact solution near the cylinder as $\nu \to 0$ and large l/a, at least far from the cylinder faces.

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DETERMINATION OF THE BOUNDARY OF A HYDRODYNAMIC CONTACT REGION

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The thickness of a lubricating film and the integral hydrodynamic contact force characteristics are determined to a significant degree by the form and dimensions of the contact region [1-5]. The present study will formulate conditions on the boundary of a planar contact region with consideration of surface tension; the problem of boundary determination is formulated within the framework of Reynolds equations.

1. Boundary Conditions for Reynolds Equations. We will consider the flow of a thin liquid layer, separating two surfaces S_1 and S_2 (Fig. 1). We denote by Ω the region within which the liquid occupies the entire interval between the surfaces. Since the layer is thin, in correspondence to Ω we will consider a surface S, lying within Ω at equal distances from S_1 and S_2. We denote by $\gamma \in S$ the boundary of the continuous liquid layer. We will consider the nonstationary problem. Let Ω, S_1, S_2, S, γ depend on time. Each point of $M \in \gamma$ can be described by a moving Cartesian coordinate system $M\xi\eta\zeta$ with unit vectors $\mathbf{n}$, $\boldsymbol{\tau}$, $\mathbf{k}$ such that the vector $\mathbf{k}$ is perpendicular to S, $\boldsymbol{\tau}$ is tangent to γ, and $\mathbf{n}$ is tangent to S and perpendicular to γ, directed outward from Ω. Let $\mathbf{u}_1$ and $\mathbf{u}_2$ be the projections of the velocities of the surfaces S_1 and S_2 on S. We will term the boundary an input ($\gamma+$), if $(\mathbf{n}, \mathbf{u}_1) \leqslant 0$, $(\mathbf{n}, \mathbf{u}_2) \leqslant 0$, $(\mathbf{n}, \mathbf{u}_1)^2 + (\mathbf{n}, \mathbf{u}_2)^2 \neq 0$, an output ($\gamma_-$), if $(\mathbf{n}, \mathbf{u}_1) \geqslant 0$, $(\mathbf{n}, \mathbf{u}_2) \geqslant 0$, $(\mathbf{n}, \mathbf{u}_1)^2 + (\mathbf{n}, \mathbf{u}_2)^2 \neq 0$, or mixed ($\gamma_\pm$), if the conditions for γ_+ and γ_- are not fulfilled. In normal applications boundaries are usually either input or output.

We will assume that the flow in Ω is described by a Reynolds equation, which requires two boundary conditions on the entire free boundary γ. Analysis of the Stokes equation near γ with consideration of surface tension on the boundary between the liquid and surrounding medium shows that if we neglect inertial terms and mass forces and assume the flow to be locally independent of coordinate η, the boundary conditions will have the following structure:

$$p = \frac{2\sigma}{h} p_+\left[\frac{\sigma}{\mu(\mathbf{u}, \mathbf{n})}, \frac{(\mathbf{u}_2 - \mathbf{u}_1, \mathbf{n})}{(\mathbf{u}, \mathbf{n})}, \frac{h_1}{h}, \frac{h_2}{h}\right]; \tag{1.1}$$

$$(\mathbf{q}_i - \mathbf{q}_0, \mathbf{n}) = 0 \tag{1.2}$$

on γ_+,

$$p = \frac{2\sigma}{h} p_\pm\left[\frac{\sigma}{\mu(\mathbf{u}_1, \mathbf{n})}, -\frac{(\mathbf{u}_2, \mathbf{n})}{(\mathbf{u}_1, \mathbf{n})}, \frac{h_1}{h}\right]; \tag{1.3}$$

$$(\mathbf{q}_i, \mathbf{n}) = h(\mathbf{u}_1, \mathbf{n})\, g\left[\frac{\sigma}{\mu(\mathbf{u}_1, \mathbf{n})}, -\frac{(\mathbf{u}_2, \mathbf{n})}{(\mathbf{u}_1, \mathbf{n})}, \frac{h_1}{h}\right] \tag{1.4}$$

on $\gamma_\pm$ at $(\mathbf{u}_1, \mathbf{n}) < 0$,

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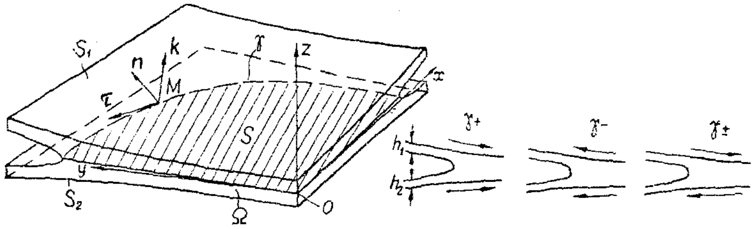

Fig. 1.

$$p = \frac{2\sigma}{h} p_{-}\left[\frac{\sigma}{\mu(\mathbf{u}, \mathbf{n})}, \frac{(\mathbf{u}_2 - \mathbf{u}_1, \mathbf{n})}{(\mathbf{u}, \mathbf{n})}\right]; \quad (1.5)$$

$$(\mathbf{q}_i, \mathbf{n}) = h(\mathbf{u}, \mathbf{n}) f\left[\frac{\sigma}{\mu(\mathbf{u}, \mathbf{n})}, \frac{(\mathbf{u}_2 - \mathbf{u}_1, \mathbf{n})}{(\mathbf{u}, \mathbf{n})}\right] \quad (1.6)$$

on γ_-. Here p is pressure, h is the gap between S_1 and S_2, $\mathbf{u} = (\mathbf{u}_1 + \mathbf{u}_2)/2$, σ is surface tension, μ is viscosity, h_1 and h_2 are the thicknesses of the liquid layers adhering to S_1 and S_2 directly, beyond the limits of Ω; $\mathbf{q}_0 = h_1\mathbf{u}_1 + h_2\mathbf{u}_2$ and $\mathbf{q}_i = -\frac{h^3}{12\mu}\left(\frac{\partial p}{\partial \xi}\mathbf{n} + \frac{\partial p}{\partial \eta}\boldsymbol{\tau}\right) + \mathbf{u}h$ are the liquid flows outside and inside Ω; p_+, p_-, $p_\pm$, f and g are functions defined by the solution of the Stokes equation. The concrete form of conditions (1.1), (1.3)-(1.6) has not been obtained at the present time. However, in cases typical of hydrodynamic contact, simplifications are possible. At characteristic pressures in Ω, significantly exceeding possible capillary pressures, Eqs. (1.1), (1.3), (1.5) can be replaced by

$$p|_\gamma = 0, \quad (1.7)$$

by including infinitely removed points within γ. Condition (1.6) for the special case $\mathbf{u}_2 = 0$ was studied approximately in [6], where it was shown that as the first argument tends to zero and infinity, f tends to unity and zero, respectively. In particular, the so-called cavitation or Reynolds boundary condition

$$\partial p/\partial \xi|_{\gamma_-} = 0 \quad (1.8)$$

can be obtained as the limit of Eq. (1.6) as $\sigma/\mu(\mathbf{u}, \mathbf{n})$ tends to zero. The authors know of no studies of conditions of the type of Eq. (1.4). We will now limit our consideration to flows with boundaries γ_+ and γ_-.

2. Formulation of the Free Boundary Determination Problem. The boundary conditions described in Sec. 1 permit formulation of the free boundary determination problem. We will make a number of simplifying assumptions. We relate a Cartesian coordinate system $Oxyz$ to the surface S, such that the plane xy is tangent to S at the point O (Fig. 1). We assume that the characteristic radius of curvature of S is significantly larger than its dimensions. Then the Reynolds equation can be considered in the coordinate system Oxy, and the boundary conditions can be imposed on the plane xy. We denote the input and output boundaries by $a(y, t)$ and $c(y, t)$. We assume that a and c are uniquely defined with respect to y and for definiteness, $c \geqslant a$.

Conditions (1.2), (1.6) can be written in the system $Oxyz$ in the form

$$(u_{1x}h_1 + u_{2x}h_2) - (u_{1y}h_1 + u_{2y}h_2)\,\partial a/\partial y - (h_1 + h_2)\,\partial a/\partial t = -\frac{h^3}{12\mu}\left(\frac{\partial p}{\partial x} - \frac{\partial p}{\partial y}\frac{\partial a}{\partial y}\right) + u_x h - u_y h\frac{\partial a}{\partial y} - h\frac{\partial a}{\partial t}; \quad (2.1)$$

$$-\frac{h^3}{12\mu}\left(\frac{\partial p}{\partial x} - \frac{\partial p}{\partial y}\frac{\partial c}{\partial y}\right) + h\left(u_x - u_y\frac{\partial c}{\partial y} - \frac{\partial c}{\partial t}\right) = h\left(u_x - u_y\frac{\partial c}{\partial y} - \frac{\partial c}{\partial t}\right) f\left[\frac{\sigma\sqrt{1 + (\partial c/\partial y)^2}}{\mu(u_x - u_y\partial c/\partial y - \partial c/\partial t)}, \frac{(u_{1x} - u_{2x}) - (u_{1y} - u_{2y})\,\partial c/\partial y}{u_x - u_y\partial c/\partial y - \partial c/\partial t}\right]. \quad (2.2)$$

Here $u_{1x}(u_{2x})$, $u_{1y}(u_{2y})$ are the projections of the velocities of the surfaces $S_1(S_2)$ on the x and y axes, $u_x = (u_{1x} + u_{2x})/2$, $u_y = (u_{1y} + u_{2y})/2$. Condition (2.1) in the special case $\partial/\partial y = 0$ and $u_{1y} = u_{2y} = 0$ gives condition (1.4) of [3]. Condition (1.7) does not change its form, while Eq. (1.8) in conjunction with Eq. (1.7) gives

$$\partial p/\partial x = 0 \quad (2.3)$$

on the boundary c(y, t) where $\partial c/\partial y$ exists.

The nonstationary Reynolds equation has the form

$$\operatorname{div}\left((h^3/12\mu)\operatorname{grad} p - \mathbf{u}h\right) = \partial h/\partial t, \tag{2.4}$$

where div and grad are taken with respect to the variables x and y. In the future we will assume that the velocities $\mathbf{u}_1$ and $\mathbf{u}_2$ do not depend on x and y and have no components along the y axis. Let h_*, u_*, l_x, l_y be characteristic values of h, $\mathbf{u}$, and the dimensions along the x and y axes. Referencing a, c, x to l_x, y to l_y, h to h_*, p to $12\mu u_* l_x h_*^{-2}$, and t to $l_x u_*^{-1}$, and maintaining the previous notation, from Eqs. (1.7), (2.1), (2.3), (2.4), we obtain a system of equations in dimensionless variables:

$$\frac{\partial}{\partial x}\left(h^3\frac{\partial p}{\partial x}\right)+\varepsilon\frac{\partial}{\partial y}\left(h^3\frac{\partial p}{\partial y}\right)=u\frac{\partial h}{\partial x}+\frac{\partial h}{\partial t},\quad p=0,\quad \frac{\partial p}{\partial y}=0 \quad \text{at} \quad x=c, \tag{2.5}$$

$$p=0,\quad u_1h_1+u_2h_2-\frac{\partial a}{\partial t}(h_1+h_2)=-h^3\left(\frac{\partial p}{\partial x}-\varepsilon\frac{\partial p}{\partial y}\frac{\partial a}{\partial y}\right)+uh-\frac{\partial a}{\partial t}h \quad \text{at} \quad x=a.$$

Here $\varepsilon = (l_x/l_y)^2$, u_1, u_2, u are the projections of $\mathbf{u}_1$, $\mathbf{u}_2$, $\mathbf{u}$ on the x axis.

3. Narrow Region Case. We will consider system (2.5) with the assumption that $h(x, y, t) = h_0(t) + h_4(x, y)$ and $\varepsilon \ll 1$. Integrating the equation over x from a to c with consideration of the boundary conditions, we obtain

$$\frac{\partial a}{\partial t}h(a, y, t)+(u_1h_1+u_2h_2)-\frac{\partial a}{\partial t}(h_1+h_2)-uh(c, y, t)-(c-a)h_0'=-\varepsilon\frac{\partial}{\partial y}\int_a^c\left(h^3\frac{\partial p}{\partial y}\right)dx. \tag{3.1}$$

Since $\varepsilon \ll 1$, we take the pressure from the solution of the one-dimensional Reynolds equation [3]:

$$p=\int_a^x\frac{u\left[h(x_1, y, t)-h(c, y, t)\right]-h_0'(x-c)}{h^3(x_1, y, t)}dx_1, \tag{3.2}$$

where h_0' is the time derivative of h_0. From Eqs. (3.1), (3.2) with use of the boundary condition p = 0 at x = c we obtain a system for definition of a and c:

$$\frac{\partial a}{\partial t}h(a, y, t)+(u_1h_1+u_2h_2)-\frac{\partial a}{\partial t}(h_1+h_2)-uh(c, y, t)- \tag{3.3}$$

$$-(c-a)h_0'=-\varepsilon\frac{\partial}{\partial y}\int_a^c\left\{h^3(x, y, t)\frac{\partial}{\partial y}\int_a^x\frac{u\left[h(x_1, y, t)-h(c, y, t)\right]-h_0'(x-c)}{h^3(x, y, t)}\times\right.$$

$$\left.\times dx_1\right\}dx,\quad \int_a^c\frac{u\left[h(x, y, t)-h(c, y, t)\right]-h_0'(x-c)}{h^3(x, y, t)}dx=0.$$

In the special case $\partial/\partial y = 0$ we obtain the result of [3]. For a given h_0' the system is parabolic with respect to a.

4. Stationary Case. In the stationary case at $u \equiv 1$, system (3.3) takes on the form

$$q-h(c, y)=-\varepsilon\frac{\partial}{\partial y}\int_a^c\left[h^3(x, y)\frac{\partial}{\partial y}\int_c^x\frac{h(x_1, y)-h(c, y)}{h^3(x_1, y)}dx_1\right]dx, \tag{4.1}$$

$$\int_a^c\frac{h(x, y)-h(c, y)}{h^3(x, y)}dx=0,$$

where $q(y) = u_1h_1(y) + u_2h_2(y)$ is the input flow. System (4.1) is an ordinary second-order differential equation in $a(y)$ (if we express c in terms of a from the second equation and substitute in the first). For a given input flow its solution may give either limited or infinite regions.

We will pose the problem of finding a closed boundary $a(y_i) = c(y_i)$ (i = 1, 2) such that at the closure points y_i we have equality to zero of the "lateral spread" flow

$$\lim_{y\to y_i}\int_a^c h^3(x, y)\frac{\partial}{\partial y}\left[\int_c^x\frac{h(x_1, y)-h(c, y)}{h^3(x, y)}dx_1\right]dx=0,\quad i=1, 2. \tag{4.2}$$

In the special case $h = h_3(y) + x^2 (h_3(y) \neq 0)$ which corresponds, for example, to a sphere in a groove or a roller on a plane, Eqs. (4.1), (4.2) take on the form

$$h(c) - q = \varepsilon \frac{d}{dy}\left\{\frac{d}{dy}\left[h_3^2 I_1\left(\frac{a}{\sqrt{h_3}}, \frac{c}{\sqrt{h_3}}\right)\right] - 3h_3 \frac{dh_3}{dy} I_2\left(\frac{a}{\sqrt{h_3}}, \frac{c}{\sqrt{h_3}}\right)\right\}; \tag{4.3}$$

$$I_0\left(\frac{a}{\sqrt{h_3}}, \frac{c}{\sqrt{h_3}}\right) = 0; \tag{4.4}$$

$$\lim_{y \to y_i}\left\{\frac{d}{dy}\left[h_3^2 I_1\left(\frac{a}{\sqrt{h_3}}, \frac{c}{\sqrt{h_3}}\right)\right] - 3h_3 \frac{dh_3}{dy} I_2\left(\frac{a}{\sqrt{h_3}}, \frac{c}{\sqrt{h_3}}\right)\right\} = 0, \tag{4.5}$$

where

$$I_0(A, C) = \int_A^C \frac{x_1^2 - C^2}{(1 + x_1^2)} dx_1; \quad I_1(A, C) = \int_A^C \frac{x_1^2 - C^2}{(1 + x_1^2)^3} \int_{x_1}^0 (1 + x_2^2)^3 dx_2 dx_1;$$

$$I_2(A, C) = \int_A^C \frac{x_1^2 - C^2}{(1 + x_1^2)^3} \int_{x_1}^0 (1 + x_2^2)^2 dx_2 dx_1.$$

Equation (4.4) gives a transcendental relationship between a and c. Using the notation $a/\sqrt{h_3} = A$, $c/\sqrt{h_3} = C$, this relationship can be approximated by the expression

$$A = -(\varkappa_0 C_0)^{-1/3}[(1 - C/C_0)^{-1/3} - 1] + C[(3C_0)^{-1}(\varkappa_0 C_0)^{-1/3} - 2], \tag{4.6}$$

where $C_0 \simeq 0.47513$ is a solution of the equation $I_0(-\infty, C_0) = 0$; $\varkappa_0 = 6C_0^2(1 + C_0^2)^{-2}(1 - 3C_0^2)^{-1} \approx 2.793$. Equation (4.6) gives correct asymptotes as $C \to 0$ and $C \to C_0$, while for $C \in (0, C_0)$ the error does not exceed 5%.

System (4.3), (4.4) contains the small parameter ε. For $\varepsilon = 0$ Eqs. (4.3), (4.4) represent the problem of determining the free boundary with linear contact. In [3] it was shown that this problem has a solution only at $q/h_3 \in [1, 1 + C_0^2]$. Thus, the limiting transition to $\varepsilon = 0$ is possible only with the indicated limitation on the input current. In the general case we assume that the interval (y_1, y_2) can be divided into segments of two types: the first ($a \to$ const as $\varepsilon \to 0$) and second ($a \to -\infty$ as $\varepsilon \to 0$). In intervals of the first type the cofactor of $\varepsilon = 0$ on the right side of Eq. (4.3) is finite and a solution can be obtained by setting $\varepsilon = 0$. In intervals of the second type Eq. (4.4) gives $c = C_0\sqrt{h_3}$, and the integrals $I_1(A, C)$ and $I_2(A, C)$ are equivalent to $A^4/28$ and $A^2/10$ as $A \to -\infty$ and Eq. (4.3) gives

$$(1 + C_0^2) h_3 - q = \frac{\varepsilon}{28} \frac{d^2(a^4)}{dy^2}. \tag{4.7}$$

The boundary conditions take on the form

$$a(y_i') = 0, \quad \lim_{y \to y_i'} \frac{d}{dy}(a^4) = 0, \quad i = 1, 2, \tag{4.8}$$

where y_i' are the boundaries of the intervals of the second type.

The boundary can be constructed in the following manner; we define those y, for which $q/h_3 > 1 + C_0^2$. Here the solution will be of the second type. The boundaries of second-type intervals can extend beyond the region where $q/h_3 > 1 + C_0^2$. They are defined in the process of solving Eqs. (4.7), (4.8). It can be shown that to several intervals with $q/h_3 > 1 + C_0^2$ there corresponds one interval of the second type solution. If $q/h_3 < 1 + C_0^2$ and the point does not fall into a second type interval, its solution is determined from the one-dimensional Reynolds equation, i.e., from system (4.3), (4.4) at $\varepsilon = 0$ (solution of the first type).

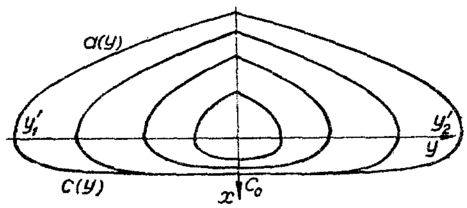

Fig. 2.

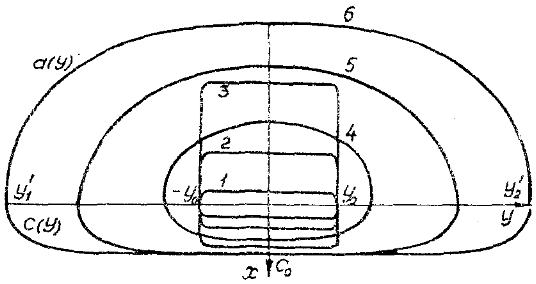

Fig. 3.

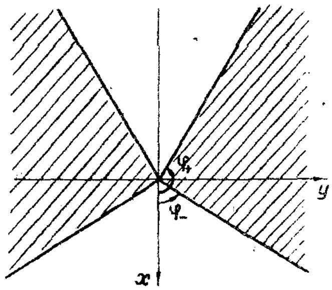

Fig. 4.

We will consider several examples. Let $h_3(y) \equiv 1$, $q(y) = q_1\delta(y)$, where δ is a delta function. Integration of Eqs. (4.7), (4.8) gives $y_i' = \pm q_1/2(1 + C_0^2)$, $a(0) = -[7q_1^2/2\varepsilon(1 + C_0^2)]^{1/4}$, and c is found from Eq. (4.4). Figure 2 depicts characteristic contact regions for various q_1. Larger q_1 correspond to a larger region.

We will consider a rectangular input flow ($q(y) = q_2$ at $|y| \leqslant y_0$, $q(y) = 0$ at $|y| > y_0$) below a cylinder $h_3(y) \equiv 1$. At $q_2 < 1 + C_0^2$ a solution of the first type is obtained. The boundaries are shown in Fig. 3 by the numbers 1-3 in order of increasing q_2. In this case $c = \sqrt{q_2 - 1}$, and a is determined from Eq. (4.4). At $q_2 > 1 + C_0^2$ a solution of the second type is obtained, and the boundaries are shown by numbers 4-6 of Fig. 3. In this case

$$a(0) = -\left[\frac{14q_2y_0^2(q_2 - 1 - C_0^2)}{\varepsilon(1 + C_0^2)}\right]^{1/4}, \quad y_i' = \pm\frac{q_2y_0}{1 + C_0^2}.$$

The dependence of $a(0)$ on q_0, as is evident from Fig. 3, is not monotonic. The point $q_2 = 1 + C_0^2$ is singular. In its vicinity the solution must be sought from the full system (4.3)-(4.5). Use of these equations after division into solutions of the first and second types can lead to large errors here.

We will consider $h_3 = 1 + y^2$, $q - 1 \ll C_0^2$, with q independent of y. This corresponds, for example, to rolling of a sphere in a lubricated channel, with slight immersion of the sphere into the lubricant. In this case a solution of the first type is obtained: $c \approx \sqrt{q - 1 - y^2}$, $a \approx -2\sqrt{q - 1 - y^2}$, $y_i \approx \pm\sqrt{q - 1}$.

The solution of Eqs. (4.3), (4.4) with the condition of equality of the distributed flows before the input and after the output boundary is of great practical interest: $h(c) = q$ in the notation of Eq. (4.3). Such a situation can obviously occur in roller bearings with stationary rotation of the roller. This condition in conjunction with the condition of symmetry with respect to y transforms Eq. (4.3) to

$$h_3\frac{d}{dy}I_1\left(\frac{a}{\sqrt{h_3}}, \frac{c}{\sqrt{h_3}}\right) = \left[3I_2\left(\frac{a}{\sqrt{h_3}}, \frac{c}{\sqrt{h_3}}\right) - 2I_2\left(\frac{a}{\sqrt{h_3}}, \frac{c}{\sqrt{h_3}}\right)\right]\frac{dh_3}{dy}. \quad (4.9)$$

Study of system (4.4), (4.9) shows that it cannot produce finite contact regions with h_3 monotonically increasing with increase in $|y|$. Apparently, it is necessary to consider surface tension to obtain such regions. With specification of various $a(0)$, solution of the system (at $h_3 = 1 + y^2$) gives a number of curves $a(y)$ and $c(y)$ which tend to common (but different for a and c) asymptotes at infinity, these being rays passing through the origin of the coordinate system. Such region forms have been observed in experiment [7].

5. Determination of the Lubricating Layer Boundaries Far from the Minimum Gap Point. We will consider the stationary flow of a lubricating layer between two contacting solid bodies as they roll. Such a situation is an idealization of real conditions, since the presence of rolling with contact leads to infinite pressures in view of the Reynolds equation. But it will describe the flow adequately at distances from the minimum gap point large in comparison to the "hydrodynamic dimension," $\sqrt{2R_{max}h_{min}}$, where R_{max} is the maximum radius of curvature of the gap form and h_{min} is the minimum film thickness.

In the Cartesian coordinate system Oxy, introduced in Sec. 2, let the gap form be described by the function $h = x^2/2R_x + y^2/2R_y$. In a polar coordinate system $Or\varphi$, the gap will have the form $h = r^2(\cos^2\varphi/2R_x + \sin^2\varphi/2R_y)$. The stationary solution of Reynolds equation (2.4) in the polar coordinate system will have the form

$$r^{-2}\frac{\partial}{\partial r}\left[r^7\frac{\partial p}{\partial r}\right](\cos^2\varphi + \varepsilon_0\sin^2\varphi)^2 + r^3\frac{\partial}{\partial\varphi}\left[(\cos^2\varphi + \varepsilon_0\sin^2\varphi)^3\frac{\partial p}{\partial\varphi}\right] = 24\,\mu u\,(2R_x)^2\cos\varphi,$$

where $\varepsilon_0 = R_x/R_y$. We assume that the input and output boundaries are rays radiating from the origin with polar angles φ_+ and φ_- (Fig. 4). Such region forms were obtained in the experiments of [8]. We will show that there exist exact solutions of the Reynolds equations describing regions of such form. For definiteness, we assume $\varphi_+ \in (\pi/2, \pi)$ $\varphi_- \in (0, \pi/2)$. We will consider solutions symmetric with respect to y, so that $2\pi - \varphi_-$ and $2\pi - \varphi_+$ are also output and input boundaries. With such assumptions boundary conditions (2.1), (2.2), (1.7) have the form

$$\frac{h^3}{12\mu\sin\varphi}\frac{1}{r}\frac{\partial p}{\partial\varphi} + uh = \begin{cases} q & \text{at } \varphi = \varphi_+ \\ huf\left(\frac{\sigma}{\mu u\sin\varphi}, \quad 0\right) & \text{at } \varphi = \varphi_-, \end{cases}$$

$$p = 0 \quad \text{at} \quad \varphi = \varphi_+, \quad \varphi = \varphi_-,$$

where $q = u(h_1 + h_2)$. The equation and boundary conditions for $q/r^2 = \text{const}$ permit seeking a solution in the form $p = \rho(r)\Phi(\varphi)$. For $\rho(r)$ we may take $\rho(r) = 12\mu u(2R_x)^2 r^{-3}$. For $\Phi(\varphi)$ we obtain the equation

$$\frac{d}{d\varphi}\left[(\cos^2\varphi + \varepsilon_0\sin^2\varphi)^3\frac{d\Phi}{d\varphi}\right] - 9\Phi(\cos^2\varphi + \varepsilon_0\sin^2\varphi)^3 = 2\cos\varphi \tag{5.1}$$

with boundary conditions

$$\frac{1}{\sin\varphi}(\cos^2\varphi + \varepsilon_0\sin^2\varphi)^2\frac{d\Phi}{d\varphi} + 1 = \begin{cases} \varkappa & \text{at } \varphi = \varphi_+; \\ f\left(\frac{\sigma}{\mu u\sin\varphi}, \quad 0\right) & \text{at } \varphi = \varphi_-; \end{cases} \tag{5.2}$$

$$\Phi(\varphi_+) = \Phi(\varphi_-) = 0, \tag{5.3}$$

where $\varkappa = 2R_x q/[u(\cos^2\varphi_+ + \varepsilon_0\sin^2\varphi_+)r^2]$.

The equation admits a particular solution

$$\Phi_0 = -[1/(3 + 2\varepsilon_0)]\cos\varphi/[(\cos^2\varphi + \varepsilon_0\sin^2\varphi)^2],$$

corresponding to the solution of [4].

For $\varepsilon_0 = 1$ Eq. (5.1) simplifies

$$d^2\Phi/d\varphi^2 - 9\Phi = 2\cos\varphi. \tag{5.4}$$

For the case of abundant lubrication, where there is no input boundary and the boundary condition at the input is replaced by a symmetry condition $\Phi(\varphi) = \Phi(2\pi - \varphi)$, the solution of Eq. (5.4) has the form

$$\Phi = -\frac{1}{5}\cos\varphi + \frac{1}{5}\frac{\cos\varphi_-\,\mathrm{ch}\,(3\varphi - 3\pi)}{\mathrm{ch}\,(3\varphi_- - 3\pi)},$$

where φ_- is found from

$$3\,\mathrm{ctg}\,\varphi_-\cdot\mathrm{th}\,(3\pi - 3\varphi_-) - 1 = 5[1 - f(\sigma/(\mu u\sin\varphi_-), 0)].$$

For $\sigma/\mu u = 0$, or $f = 1$, we obtain $\varphi_- = 1.239$. For $\sigma/\mu u = \infty$ or $f = 0$, we obtain $\varphi_- = 0.4636$. Since in the given case φ_- cannot be larger than 1.249, and at such φ_- the quantity $\mathrm{th}(3\pi - 3\varphi_-)$ is close to 1, then with good accuracy ($\sim 10^{-5}$) to define φ_- we can recommend the expression

$$3\,\mathrm{ctg}\,\varphi_- - 1 = 5[1 - f(\sigma/(\mu u\sin\varphi_-), 0)].$$

It can be concluded that with abundant lubrication the angle φ_- lies in the range from 0.4636 to 1.249 (or 26.2° to 71.6°).

In the case of insufficient lubrication (i.e., in the presence of an input boundary) at $\varepsilon_0 = 1$

$$\Phi = -(1/5)\cos\varphi + (\cos\varphi_-\mathrm{sh}\,(3\varphi_+ - 3\varphi) + \cos\varphi_+\mathrm{sh}\,(3\varphi - 3\varphi_-))/(5\mathrm{sh}(3\varphi_+ - 3\varphi_-)).$$

From condition (5.2) at the output it follows that

$$2 + \frac{\cos\varphi_+ - \cos\varphi_-\,\mathrm{ch}\,(3\varphi_+ - 3\varphi_-)}{\sin\varphi_-\,\mathrm{sh}\,(3\varphi_+ - 3\varphi_-)} = \frac{5}{3}f\left(\frac{\sigma}{\mu u\sin\varphi_-}, \quad 0\right).$$

The "periodicity" condition (equality of flow at input $q_+(y)$ to flow at output $q_-(y)$) is of

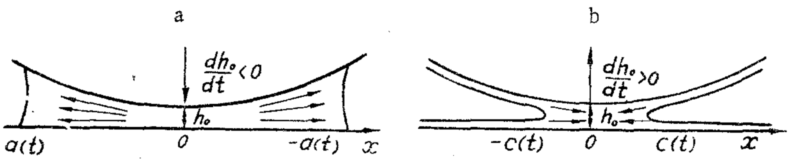

Fig. 5.

interest:

$$\frac{1}{\sin^2\varphi_+}\left[\frac{1}{\sin\varphi_+}\frac{d\Phi(\varphi_+)}{d\varphi}+1\right]=\frac{1}{\sin^2\varphi_-}\left[\frac{1}{\sin\varphi_-}\frac{d\Phi(\varphi_-)}{d\varphi}+1\right].$$

6. Limited Lubrication in Approach and Separation of Rigid Cylinders. We will consider rigid cylinders $h = h_0(t) + x^2/2R$. Equation (2.4) at $u \equiv 0$ and $\partial/\partial y = 0$ has the form

$$\frac{\partial}{\partial x}\left(h^3\frac{\partial p}{\partial x}\right)=12\mu\frac{\partial h}{\partial t}. \tag{6.1}$$

Now let the flow be symmetric about the point $x = 0$.

We will consider the "crushing" of an oil droplet (Fig. 5a). In this case the boundaries $a(t)$ and $-a(t)$ are input boundaries. Condition (2.1) reduces to

$$\frac{h^3}{12\mu}\frac{\partial p}{\partial x}+h\frac{\partial a}{\partial t}=0 \tag{6.2}$$

at $x = a$. Eliminating p from Eqs. (6.1), (6.2), and (1.7) gives

$$-h_0a - a^3/6R = V/2, \tag{6.3}$$

where V is the liquid volume per unit cylinder length. Equation (6.3) expresses the law of conservation of volume. For the pressure we obtain

$$p = -(6R\mu dh_0/dt)[1/h^2 - 1/h^2(a)]. \tag{6.4}$$

We will consider separation of the cylinders (Fig. 5b). Now the boundaries c(t) and $-c(t)$ are outputs. Condition (2.2) takes on the form

$$\frac{h^3}{12\mu}\frac{\partial p}{\partial x}+h\frac{dc}{dt}=\frac{dc}{dt}hf\left(-\frac{\sigma}{\mu dc/dt},\quad 0\right). \tag{6.5}$$

The solution of Eqs. (6.1), (6.5), (1.7) at $\sigma = \infty (f = 0)$ again gives the law of conservation of volume $h_0c + c^3/6R = V/2$, and the pressure is determined from Eq. (6.4) with a replaced by c. If $\sigma < \infty$, as is shown in Fig. 5b, a certain layer of liquid will remain on the walls, the volume of liquid in contact $x \in (-c, c)$ decreasing. Application of the cavitation condition $\sigma = 0 (f = 1)$ in the given case leads to instantaneous breakoff of the lubricating film, i.e., surface tension cannot be neglected in the problem of separation.

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